Models of Set Theory II - Winter 2017/2018

Prof. Dr. Peter Koepke, Ana Njegomir Problem sheet 8

Problem 1 (2 points)**.** Let T be a Suslin tree and $\mathbb{P}_T$ be the corresponding forcing notion. Show that $\mathbb{P}_T \times \mathbb{P}_T$ is not ccc.

Definition. For $x, y \subseteq \omega$ we say that x is *almost contained* in y, denoted $x \subseteq^* y$, if $x \setminus y$ is finite. A *pseudo-intersection* of a family $\mathcal{F} \subseteq [\omega]^\omega$ is an element $x \in [\omega]^\omega$ such that for every $y \in \mathcal{F}$, $x \subseteq^* y$. Furthermore, we say that $\mathcal{F} \subseteq [\omega]^\omega$ has the *strong finite intersection property (sfip)*, if every finite subfamily of $\mathcal{F}$ has infinite intersection.
The *pseudo-intersection number* $\mathfrak{p}$ is defined as the least cardinality of a family $\mathcal{F} \subseteq [\omega]^\omega$ which has the sfip but does not have a pseudo-intersection.

Problem 2 (2 points)**.** Prove that $\aleph_1 \leq \mathfrak{p}$.

Problem 3 (5 points)**.** Let $\mathcal{F}, \mathcal{G} \subseteq [\omega]^\omega$ be nonempty families of size $< \mathfrak{p}$ such that for all $y \in \mathcal{G}$, $\{x \cap y \mid x \in \mathcal{F}\}$ has the sfip.

(a) Let $\mathcal{F}^* = \{\bar{x} \mid x \in \mathcal{F}\} \cup \{\tilde{y} \mid y \in \mathcal{G}\} \cup \{z_n \mid n \in \omega\}$, where for $x \in \mathcal{F}, y \in \mathcal{G}$ and $n \in \omega$, $\bar{x} = \{s \in [\omega]^{<\omega} \mid s \subseteq x\}, \tilde{y} = \{s \in [\omega]^{<\omega} \mid s \cap y \neq \emptyset\}$ and $z_n = \{s \in [\omega]^{<\omega} \mid \min s > n\}$. Show that $\mathcal{F}^*$ has the sfip.

(b) Show that $\mathcal{F}$ has a pseudo-intersection x such that for each $y \in \mathcal{G}$, $x \cap y$ is infinite.

Problem 4 (2 points)**.** Let $\{I_n \mid n \in \omega\}$ be an enumeration of all open intervals in $\mathbb{R}$ with rational endpoints. Suppose that $\{D_\alpha \mid \alpha < \kappa\}$ is a set of dense open subsets of $\mathbb{R}$. Let $x_\alpha = \{n \in \omega \mid I_n \subseteq D_\alpha\}$ for $\alpha < \kappa$ and $y_k = \{n \in \omega \mid I_n \subseteq I_k\}$ for $k \in \omega$. Show that for each $k \in \omega$, $\{x_\alpha \cap y_k \mid \alpha < \kappa\}$ has the sfip.

Problem 5 (4 points)**.** Show that $\mathfrak{p} \leq \mathrm{add}(\mathcal{M})$.

Problem 6 (5 points)**.** Let M be a ground model of ZFC+CH, and let $M \models$ κ is a regular cardinal $> \aleph_1$. Let $M[G]$ be a generic extension of M by the partial order for adjoining κ Cohen reals using finite conditions. Then, in $M[G]$, $\mathfrak{b} = \aleph_1$ and $\mathfrak{d} = 2^{\aleph_0}$.

Please hand in your solutions on Monday, December 4 before the lecture.